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Relativistic theory of magnetoelastic interactions II. Constitutive theory

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Abstract. In this part of a work devoted to the study of relativistic continua which exhibit gyromagnetic phenomena, constitutive equations are obtained for a hyperelastic solid by the use of the thermodynamical admissibility (Clausius–Duhem inequality), the relevant potential being the magneto-free energy written in an *ad hoc* invariant form after application of the author's formulation of the principle of material frame indifference. The agreement with equations derived before from a variational formulation is shown.

1. Introduction

Part II of the present work is a continuation to part I (Maugin 1972f)† in which we set forth the field equations of a relativistically invariant theory of magnetoelastic interactions where the magnetic spin is taken into account. In contrast to the variational treatment given by Maugin and Eringen (1972c), these equations were established for general thermodynamical processes. We may have dissipative stresses, heat propagation and relaxation of the magnetic spin. It is the purpose of this paper to develop with the aid of the principles of thermodynamics, constitutive equations for these general processes. That is, this work constitutes the relativistic counterpart of the classical three-dimensional theory developed by Maugin and Eringen (1972a, b) and Maugin (1972h) (see also Maugin 1972g). We recall that the present problem resorts to *micromagnetic* theory whose main concern is the study of the time evolution of the magnetization vector also referred to as magnetic spin. This constitutes the main improvement with respect to the theory of magnetoelastic interactions achieved by Grot and Eringen (1966).

Using the notations of Maugin (1972f), ρ being the relativistically invariant matter density and $\mathscr{M}_\alpha$ the magnetization four vector per unit of proper volume, the magnetization four vector per unit of proper mass is defined as

$$\tilde{\mathscr{M}}_\alpha = \frac{\mathscr{M}_\alpha}{\rho}, \qquad \tilde{\mathscr{M}}^\alpha u_\alpha = 0. \tag{1.1}$$

The instantaneous motion of this *essentially spatial axial* four vector is characterized by the infinitesimal Lorentz transformation which defines the evolution of the system as the time goes on. Hence the proper time rate of $\tilde{\mathscr{M}}^\alpha$ can be written

$$\dot{\tilde{\mathscr{M}}}^\alpha = -\Xi^{\alpha}_{.\beta}\tilde{\mathscr{M}}^\beta \tag{1.2}$$

† Equations of part I are referred to with a prefix I, for example, (I-3.26) is equation (3.26) of part I.

where $\Xi_{\alpha\beta}$ is a skewsymmetric 4×4 matrix whose general covariant decomposition is (compare equations (I-2.15) and (I-2.16)),

$$\left.\begin{aligned} &\Xi_{\alpha\beta} = \Omega_{\alpha\beta}+\frac{2}{c}\zeta_{[\alpha}u_{\beta]} \\ &\Omega_{\alpha\beta} \equiv P_{\alpha}^{\cdot\gamma}P_{\beta}^{\cdot\delta}\Xi_{\gamma\delta}, \qquad \Omega_{\alpha\beta} = -\Omega_{\beta\alpha}, \qquad \Omega_{\alpha\beta}u^{\beta} = 0 \\ &\zeta_{\alpha} = -\frac{1}{c}\Xi_{\alpha\beta}u^{\beta}, \qquad \zeta_{\alpha}u^{\alpha} = 0 \end{aligned}\right\} \tag{1.3}$$

where u^{α} and $P_{\alpha\beta}$ are the four velocity and the projection operator (cf I). $\Omega_{\alpha\beta}$, the *rotation tensor* of $\tilde{\mathcal{M}}^{\alpha}$, and ζ_{α} are PU tensor fields (cf I). The value of ζ_{α} is determined by noting that $\tilde{\mathcal{M}}^{\alpha}$ is also PU. Thus, taking the inner product of equation (1.2) with u^{α}, one gets for every four vector $\tilde{\mathcal{M}}^{\beta}$

$$\tilde{\mathcal{M}}^{\beta}(\dot{u}_{\beta}-\Xi^{\alpha}_{\cdot\beta}u_{\alpha}) = 0. \tag{1.4}$$

Hence, after equation (1.3 part five)

$$\zeta_{\beta} = \frac{1}{c}\dot{u}_{\beta}. \tag{1.5}$$

Then, equation (1.2) reads

$$\dot{\tilde{\mathcal{M}}}_{\alpha} = \Omega_{\beta\alpha}\tilde{\mathcal{M}}^{\beta}+\frac{1}{c^2}\dot{u}_{\beta}u_{\alpha}\tilde{\mathcal{M}}^{\beta}. \tag{1.6}$$

Clearly, the second term represents a Fermi–Walker transport, that is, the four vector $\tilde{\mathcal{M}}^{\beta}$ is carried along with the motion x^{α} while it rotates about the event point x^{α} by an amount given by the rotation velocity $\Omega_{\alpha\beta}$. Notice that equation (1.6) is similar to the equation of time evolution of a *director* in the theory of *oriented relativistic continuous media* (cf Maugin and Eringen 1972d). Hence $\tilde{\mathcal{M}}_{\alpha}$ plays the role of a *director* while the tensor $\Omega_{\alpha\beta}$ replaces the gyration tensor $\nu_{\beta\alpha}$ of relativistic micropolar media theory. This analogy was already noticed in the classical three-dimensional theories (cf Maugin and Eringen 1972a).

Equation (1.6) is also verified for *saturated magnetization* since, taking the inner product of (1.6) with $\tilde{\mathcal{M}}^{\alpha}$, one gets from the skewsymmetry of $\Omega_{\beta\alpha}$ and the PU character of $\tilde{\mathcal{M}}_{\alpha}$

$$\dot{\tilde{\mathcal{M}}}_{\alpha}\tilde{\mathcal{M}}^{\alpha} = 0, \qquad \tilde{\mathcal{M}}_{\alpha}\tilde{\mathcal{M}}^{\alpha} = \text{constant} \tag{1.7}$$

along a world line.

Finally, introducing in a unique way π^{μ}, the dual of $\Omega_{\alpha\beta}$ in M^4, by

$$\begin{aligned} &\Omega_{\beta\alpha} \equiv \frac{1}{ic}\epsilon_{\beta\alpha\mu\nu}\pi^{\mu}u^{\nu}, \qquad \pi^{\mu} = \frac{1}{2ic}\epsilon^{\mu\nu\rho\sigma}\Omega_{\nu\rho}u_{\sigma} \\ &\pi^{\mu}u_{\mu} = 0 \end{aligned} \tag{1.8}$$

equation (1.6) reads

$$\dot{\tilde{\mathcal{M}}}_{\alpha} = \frac{1}{ic}\epsilon_{\alpha\mu\beta\nu}\tilde{\mathcal{M}}^{\beta}\pi^{\mu}u^{\nu}+\frac{1}{c^2}\dot{u}_{\beta}u_{\alpha}\tilde{\mathcal{M}}^{\beta}. \tag{1.9}$$

For small velocities, this yields the classical spin equation

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\pi} \times \boldsymbol{\mu} + O(c^{-2}) \tag{1.10}$$

in which $\boldsymbol{\mu}$ is the three-dimensional magnetization vector per unit mass and $\boldsymbol{\pi}$ is its angular velocity (cf Maugin and Eringen 1972a).

In résumé, the present problem, apart from the study of the different terms which contribute to the energy–momentum tensor, is to find the form of the tensor $\Omega_{\beta\alpha}$ (equivalently π^{μ}) as a function of the phenomenological representation of the different interactions that take place in the medium.

Following the results of part I, the equation of conservation of internal energy and that of entropy production are examined in § 2. Constitutive equations for conservative processes are established in § 3 for nonlinear elastic materials (ie subject to large deformation fields) by use of the Clausius–Duhem inequality. This follows the study of Lorentz invariance and the application of the principle of objectivity to the magneto-free energy function. Some remarks concerning the saturation of the magnetization are made in § 4. The results so obtained are brought together with those of the variational approach given previously (Maugin and Eringen 1972c), in § 5. Dissipative processes which lead to heat propagation and relaxation of the spin will be examined in part III following the use of Ziegler's (1963) principle of least irreversible force, representation theorems and Onsager's relations.

2. Thermodynamical equations

In this section, we give a form of the second principle of thermodynamics appropriate to the study of constitutive equations. The study of general thermodynamical processes in continuum physics is based on three equations or inequalities. The first two of these are the equation of conservation of energy—equivalent to the statement of the first principle of thermodynamics—and the entropy inequality—equivalent to the statement of the second principle of thermodynamics. In the present theory, these equations have been given in part I†

$$\rho\dot{\epsilon} + q^{\beta}_{;\beta} + p^{\alpha}\dot{u}_{\alpha} - t^{\beta\alpha}u_{\alpha;\beta} = -\rho h + \mathscr{E}_{\gamma}j^{\gamma} - \rho\tilde{\mathscr{M}}^{\alpha}\dot{\mathscr{B}}_{\alpha} - \frac{1}{c}\mathscr{E}_{\alpha}\overset{*}{\Pi}{}^{\beta\alpha}\dot{u}_{\beta} \tag{2.1}$$

$$\rho\dot{\eta} + \frac{1}{\theta}\hat{q}^{\alpha}_{;\alpha} - \frac{1}{\theta^{2}}\hat{q}^{\alpha}\theta_{,\alpha} + \frac{\rho h}{\theta} \geqslant 0. \tag{2.2}$$

We recall the notation $\epsilon, \eta, \theta, h, \hat{q}^{\alpha}, t^{\beta\alpha}, \mathscr{B}_{\alpha}$ and $\mathscr{E}_{\alpha}$ which are the specific internal energy, the specific entropy, the thermodynamical temperature, the heat source per unit of proper mass, the heat flux four vector‡, the relativistic stress tensor, the magnetic intensity four vector and the electric four vector respectively. j^{γ} is the conduction current. A superposed dot denotes proper time differentiation. $\overset{*}{\Pi}{}^{\beta\alpha}$ is the quantity defined as

$$\overset{*}{\Pi}{}^{\beta\alpha} = \frac{1}{\mathrm{i}c}\epsilon^{\beta\alpha\mu\nu}\mathscr{M}_{\mu}u_{\nu}. \tag{2.3}$$

† A minus sign is missing from the term ρh in equation (I-4.8) while equation (I-3.32 part four) should read $h = f^{\alpha}u_{\alpha}$.

‡ The quantity q^{β} appearing in equation (2.1) is only a notation in the decomposition of the stress–energy–momentum tensor. In the present theory, it differs from the heat flux vector $\hat{q}^{\beta}$ by a term due to the presence of exchange forces (compare hereafter).

The form of p^α, the so-called *nonmechanical momentum*, has been given in part I†:

$$p^\beta = \frac{1}{c^2}(q^\beta + \rho S^{\beta\alpha}\dot{u}_\alpha + 2M^{\alpha\beta\gamma}u_{\alpha;\gamma}) - \frac{1}{ic^2}\epsilon^{\beta\gamma\mu\nu}u_\nu \mathscr{M}_\mu \mathscr{E}_\gamma \tag{2.4}$$

in which $S^{\beta\alpha}$ is the *intrinsic spin* and $M^{\alpha\beta\gamma}$ is the *relativistic couple stress tensor.*

The third equation, in fact equivalent to equation (2.2), is the entropy production equation written as follows. Introducing the specific free (Helmholtz) function Ψ by the definition

$$\Psi = \epsilon - \eta\theta \tag{2.5}$$

and differentiating with respect to proper time, one obtains

$$\rho\dot{\eta} = \frac{\rho\dot{\epsilon}}{\theta} - \frac{\rho}{\theta}(\dot{\Psi} + \eta\dot{\theta}). \tag{2.6}$$

Other forms can be given to equations (2.2) and (2.6). The term defined as

$$\mathscr{C} = -\left\{\left(\frac{\hat{q}^\alpha}{\theta}\right)_{;\alpha} + \frac{\rho h}{\theta}\right\} \tag{2.7}$$

is called the *Clausius term* (cf Maugin 1971c). Then equation (2.2) reads

$$\rho\dot{\eta} - \mathscr{C} \geqslant 0. \tag{2.8}$$

The positive quantity so defined we call the *Jouguet term*

$$\rho\dot{\eta} - \mathscr{C} = \mathscr{J} \equiv \rho^{\mathrm{R}}\dot{\eta} + \rho\phi \tag{2.9}$$

in which ϕ is the *dissipation function density*. The recoverable entropy rate ${}^{\mathrm{R}}\dot{\eta}$ will be shown to vanish in § 3.3. Thus we will have

$$\Phi = \rho\theta\phi \geqslant 0 \tag{2.10}$$

which is the dissipation inequality. The form of Φ is given in § 3.3.

It is interesting to use other variables in lieu of ϵ and Ψ. In fact, the internal energy ϵ is supposed to be dependent on the field $\mathscr{B}_\alpha$. We prefer to use the magneto-internal energy e which depends on the magnetization vector (cf equation (I-3.43) and Fokker 1939)

$$e(\tilde{\mathscr{M}}^\gamma) = \epsilon(\mathscr{B}^\gamma) + \tilde{\mathscr{M}}^\alpha \mathscr{B}_\alpha. \tag{2.11}$$

Accordingly, the magneto-free energy Ψ^* is defined as (cf Maugin 1972a, e)

$$\Psi^* = e - \eta\theta = \Psi + \tilde{\mathscr{M}}^\alpha \mathscr{B}_\alpha. \tag{2.12}$$

It follows that equation (2.1) is written

$$\rho\dot{e} + q^\beta{}_{;\beta} + p^\alpha\dot{u}_\alpha - t^{\beta\alpha}u_{\alpha;\beta} = -\rho h + \mathscr{E}_\gamma j^\gamma + \rho\mathscr{B}_\alpha\dot{\tilde{\mathscr{M}}}^\alpha - \frac{1}{ic^2}\epsilon^{\beta\alpha\mu\nu}\mathscr{M}_\mu\mathscr{E}_\alpha u_\nu \dot{u}_\beta \tag{2.13}$$

and equation (2.6) reads

$$\rho\dot{\eta} = \frac{\rho\dot{e}}{\theta} - \frac{\rho}{\theta}(\dot{\Psi}^* + \eta\dot{\theta}). \tag{2.14}$$

† It should be a factor of $1/c^2$ instead of $1/c$ in the second and third terms of the right hand side of equation (I-4.6).

Further transformations of the inequality (2.2) with the aid of equations (2.13) and (2.14) will lead to the useful Clausius–Duhem inequality. Before, we transform equation (2.13), we carry the value of p^β provided by equation (2.4) into (2.13) and use equation (1.6). We thus obtain

$$\rho\dot{e}+q^\beta{}_{;\beta}+\frac{1}{c^2}q^\beta\dot{u}_\beta-t^{\beta\alpha}u_{\alpha;\beta}+\frac{2}{c^2}M^{\alpha\beta\gamma}u_{\alpha;\gamma}\dot{u}_\beta = -\rho h+\mathscr{E}_\gamma j^\gamma+\rho\mathscr{B}^\alpha\tilde{\mathscr{M}}^\beta\Omega_{\beta\alpha} \tag{2.15}$$

where we used the skewsymmetry of $S^{\beta\alpha}$ and the fact that $\mathscr{B}_\alpha$ is PU. Furthermore, since $t^{\beta\alpha}$ is PU and using the decomposition of $e_{\alpha\beta}$ (cf equations (I-2.20))

$$e_{\alpha\beta} \equiv P_\beta^{\cdot\mu}\nabla_\mu u_\gamma P^{\gamma}_{\cdot\alpha} = \sigma_{\alpha\beta}+\omega_{\alpha\beta} \tag{2.16}$$

we have

$$t^{\beta\alpha}u_{\alpha;\beta} = t^{(\beta\alpha)}\sigma_{\alpha\beta}+t^{[\beta\alpha]}\omega_{\alpha\beta}. \tag{2.17}$$

Notice that

$$\rho\mathscr{B}^\alpha\tilde{\mathscr{M}}^\beta\Omega_{\beta\alpha} = \rho\mathscr{B}^{[\alpha}\tilde{\mathscr{M}}^{\beta]}\Omega_{\beta\alpha}. \tag{2.18}$$

The value of $\rho\mathscr{B}^{[\alpha}\tilde{\mathscr{M}}^{\beta]}$ is given by an equation derived in part I (cf equation (I-4.5))

$$\rho\mathscr{B}^{[\alpha}\tilde{\mathscr{M}}^{\beta]} = -\frac{\rho}{2}\dot{S}^{\alpha\beta}+t^{[\alpha\beta]}+M^{\alpha\beta\gamma}{}_{;\gamma}+\frac{1}{c^2}\rho S^{[\alpha}_{\cdot\gamma}u^{\beta]}\dot{u}^\gamma-\frac{2}{c^2}M^{[\alpha|\gamma\mu|}u^{\beta]}u_{\gamma;\mu}. \tag{2.19}$$

We need to compute the quantity

$$\rho\mathscr{B}^{[\alpha}\tilde{\mathscr{M}}^{\beta]}\Omega_{\beta\alpha} = -\frac{\rho}{2}\dot{S}^{\alpha\beta}\Omega_{\beta\alpha}+t^{[\alpha\beta]}\Omega_{\beta\alpha}+\frac{1}{c^2}\rho S^{[\alpha}_{\cdot\gamma}u^{\beta]}\dot{u}^\gamma\Omega_{\beta\alpha}+M^{\alpha\beta\gamma}{}_{;\gamma}\Omega_{\beta\alpha}-\frac{2}{c^2}M^{[\alpha|\gamma\mu|}u^{\beta]}u_{\gamma;\mu}\Omega_{\beta\alpha}. \tag{2.20}$$

The third and last term in the right hand side vanish because of the PU character of $\Omega_{\beta\alpha}$. The first term vanishes since the magnetic spin yields a *d'Alembertian inertia couple*, that is, $\dot{S}^{\alpha\beta}$ does not work in a real rotation $\Omega_{\beta\alpha}$ (cf appendix 1). The fourth term can be written as

$$M^{\alpha\beta\gamma}{}_{;\gamma}\Omega_{\beta\alpha} = (M^{\alpha\beta\gamma}\Omega_{\beta\alpha})_{;\gamma}-M^{\alpha\beta\gamma}\Omega_{\beta\alpha;\gamma}. \tag{2.21}$$

At this point it is convenient to set†

$$q^\beta = \hat{q}^\beta+M^{\mu\nu\beta}\Omega_{\nu\mu}. \tag{2.22}$$

Carrying the expressions (2.17), (2.20) and (2.22) into equation (2.15), we obtain the equation of conservation of energy in the form

$$\rho\dot{e}+\hat{q}^\beta{}_{;\beta}+\frac{1}{c^2}\hat{q}^\beta\dot{u}_\beta-t^{(\beta\alpha)}\sigma_{\alpha\beta}-t^{[\beta\alpha]}(\omega_{\alpha\beta}+\Omega_{\alpha\beta})-M^{\alpha\beta\gamma}A_{\beta\alpha\gamma}-\mathscr{E}_\gamma j^\gamma = -\rho h \tag{2.23}$$

in which we have defined the kinematical quantity

$$A_{\beta\alpha\gamma} = -\left(\Omega_{\beta\alpha;\gamma}+\frac{1}{c^2}\Omega_{\beta\alpha}\dot{u}_\gamma-\frac{2}{c^2}u_{[\beta;|\gamma|}\dot{u}_{\alpha]}\right). \tag{2.24}$$

Now, following the usual method, we eliminate the heat supply between equations

† This is in conformity with the liberty left when one establishes a model of magnetomechanical interactions (so far we are concerned with the form of the stress–energy–momentum); in this respect, see Sedov (1965).

(2.23) and (2.2) and use equation (2.14) to get the Clausius–Duhem inequality in the form we shall keep from here on

$$-\frac{\rho}{\theta}(\dot{\Psi}^* + \eta\dot{\theta}) - \frac{1}{\theta^2}\hat{q}^\beta \overset{*}{\theta}_\beta + \frac{\mathscr{E}_\gamma j^\gamma}{\theta} + \frac{1}{\theta}t^{(\beta\alpha)}\sigma_{\alpha\beta} + \frac{1}{\theta}t^{[\beta\alpha]}(\omega_{\alpha\beta} + \Omega_{\alpha\beta}) + \frac{1}{\theta}M^{\alpha\beta\gamma}A_{\beta\alpha\gamma} \geqslant 0 \tag{2.25}$$

in which we introduced the relativistic temperature gradient (cf Maugin 1971c)

$$\overset{*}{\theta}_\beta = P^{\cdot\gamma}_{\beta}(\theta_{,\gamma} + c^{-2}\theta\dot{u}_\gamma) \tag{2.26}$$

since $\hat{q}^\beta$ is PU.

In the next section, we establish, for a special class of material, a reduced form of Ψ^* from which we shall compute the proper time rate $\dot{\Psi}^*$ needed in equation (2.25).

3. Conservative processes

3.1. The Lorentz invariance requirement

In this section, the recoverable parts of the constitutive equations will be shown to be derivable from a potential, the relativistic magneto-free energy density Ψ^*. Before doing so, the physical behaviour of the material must be specified. In the following, we consider a *heat conducting nonlinear elastic solid of grade one* in interaction with the magnetic field. Furthermore the solid may be an electric conductor. The material symmetry (ie the crystallographic group to which the material belongs) is not specified. In order to represent the elastic behaviour of a solid of grade one, we need to introduce only the first relativistic gradient of the motion $x^{\alpha}_{\cdot K}$ as an argument of Ψ^* (cf Maugin 1972d). In order to take account of a phenomenological representation of anisotropy and exchange force fields observed in ferromagnetic materials, one must consider a dependence on the magnetization four vector and its *ad hoc* constructed gradient (cf Maugin and Eringen 1972a, c). Finally heat conduction leads to a dependence of Ψ^* on θ and the relativistic temperature gradient $\overset{*}{\theta}_\alpha$ defined by equation (2.26). Material inhomogeneity implies an explicit dependence of Ψ^* upon the lagrangian coordinates X^K. Hence we consider

$$\Psi^* = \Psi^*(x^{\alpha}_{\cdot K}; \tilde{\mathscr{M}}_\alpha; M^{\alpha}_{\cdot K}; \mathscr{E}^\alpha; \theta; \overset{*}{\theta}_\alpha; X^K). \tag{3.1}$$

A material defined by equation (3.1) presents no hereditary character for Ψ^* is a classical function of class C^m—at least of class C^1—with respect to its arguments and not a functional. The value of the arguments is taken at the present event point x^α of the space–time manifold of Minkowski M^4. It is easily verified that, in agreement with the postulates set forth by Maugin (1971b, 1972f), all these arguments are PU four vector fields or proper invariant (such as θ and X^K) since we have defined

$$x^{\alpha}_{\cdot K} \equiv P^{\alpha}_{\cdot\beta}x^{\beta}_{,K}, \qquad x^{\alpha}_{\cdot K}u_\alpha = 0 \tag{3.2}$$

$$M^{\alpha}_{\cdot K} \equiv P^{\alpha}_{\cdot\gamma}\tilde{\mathscr{M}}^{\gamma}_{\cdot;\mu}x^{\mu}_{\cdot K}, \qquad M^{\alpha}_{\cdot K}u_\alpha = 0 \tag{3.3}$$

$$\overset{*}{\theta}_\alpha u^\alpha = 0, \qquad \mathscr{E}^\alpha u_\alpha = 0. \tag{3.4}$$

Finally, we recall that Ψ^* is a *proper scalar*, that is, measured by an observer following the element of continuum in its motion. Therefore it is a Lorentz invariant. This brings a constraint to the function form of Ψ^*. Indeed, considering an infinitesimal transformation of the form (space–time translations need not be considered since Ψ^* which does not depend explicitly on x^α is obviously form invariant under such transformations)

$$x^{*\alpha} = \epsilon Q^{\alpha}_{.\beta} x^{\beta} \tag{3.5}$$

in which ϵ is infinitesimally small and $Q^{\alpha\beta}$ is skewsymmetric, the invariance of Ψ^* under such a transformation yields for arbitrary $Q^{\alpha\beta}$

$$\frac{\partial \Psi^*}{\partial x^{[\alpha}_{.K}} x_{\beta]K} + \frac{\partial \Psi^*}{\partial \tilde{\mathscr{M}}^{[\alpha}} \tilde{\mathscr{M}}_{\beta]} + \frac{\partial \Psi^*}{\partial M^{[\alpha}_{.K}} M_{\beta]K} + \frac{\partial \Psi^*}{\partial \mathscr{E}^{[\alpha}} \mathscr{E}_{\beta]} + \frac{\partial \Psi^*}{\partial \overset{*}{\theta}{}^{[\alpha}} \overset{*}{\theta}_{\beta]} = 0. \tag{3.6}$$

This linear first order partial differential equation can be integrated by the usual method (characteristics). The solution obviously depends upon Lorentz invariants—here, material tensors (in other words, lagrangian measures). The integral of equation (3.6) is

$$\Psi^* = \Psi^*(C_{KL}, M_L, M_{LK}, \mathscr{E}_K, \theta, \theta_K, X^K) \tag{3.7}$$

where the Green deformation tensor C_{KL} and the other lagrangian measures are defined by

$$C_{KL} \equiv x_{\alpha K} x^{\alpha}_{.L}, \qquad M_L \equiv x^{\alpha}_{.L} \tilde{\mathscr{M}}_{\alpha} \tag{3.8}$$

$$M_{LK} \equiv x^{\alpha}_{.L} M_{\alpha K} = x^{\gamma}_{.L} \tilde{\mathscr{M}}_{\gamma;\mu} x^{\mu}_{.K} \tag{3.9}$$

$$\theta_K \equiv x^{\alpha}_{.K} \overset{*}{\theta}_{\alpha} = x^{\alpha}_{.K}(\theta_{,\alpha} + c^{-2}\theta \dot{u}_{\alpha}), \qquad \mathscr{E}_K \equiv x^{\alpha}_{.K} \mathscr{E}_{\alpha}. \tag{3.10}$$

As was already emphasized by Kafadar and Eringen (1971) and Maugin and Eringen (1972c, d), it is remarkable that the function dependence (3.7) obtained by application of the Lorentz invariance imposed to Ψ^* is, for materials whose basic mechanical behaviour is *elastic*, identical to that obtained by application of all presently formulated forms of the so-called *principle of objectivity* in relativistic continuum mechanics. This question is briefly examined in the next section.

3.2. *Objectivity*

The very controversial question of constructing a relativistic extension of the principle of objectivity (principle of material frame indifference) of classical continuum mechanics has been approached from many viewpoints. Whereas Bressan (1963) suggests using directly lagrangian measures in the constitutive variables without any further reference to any such principle (a procedure used by Maugin (1971d) in his study of magnetized media in general relativity), Bragg (1965) postulates his principle of nonsentient response, and Söderholm (1970, 1971), by studying classes of so-called equivalent motions in special relativity, gives a formulation very close to that of classical continuum mechanics. The latter formulation has been used by Maugin and Eringen in their studies on polarized elastic materials (Maugin and Eringen 1972c) and oriented elastic materials (Maugin and Eringen 1972d). Grot and Eringen (1966) and, in fact, Kafadar and Eringen (1971) prefer to consider the invariance under the *constant* group of homogeneous Lorentz transformations. In that case, the remark made at the end of the foregoing section is supererogatory. More recently, the present author has proposed, in the frame of general relativity theory, to replace a pure extension of the *énoncé* of the classical notion of objectivity by a postulate on the functional dependence of the constitutive variables (Maugin 1972b). These variables should depend only on quantities measured in the local Fermi frame. This postulate has been applied with success to different media presenting hereditary properties (Maugin 1972c). Moreover, such a formulation encompasses, in special relativity, that of Söderholm which we consider to be the best one in special relativistic continuum mechanics (cf Maugin to be published). In any case, one must note that the objectivity principle takes all its importance in the study of constitutive *functionals*. For the case of constitutive *functions*, the Lorentz invariance is indeed sufficient in special relativity.

The application of Maugin's (1972b) formulation goes as follows. Call $x^{\bar{\alpha}}$, $\bar{\alpha} = 1, 2, 3, 4$, a local chart in the neighbourhood of event point $M: x^{\alpha} = \mathscr{X}^{\alpha}(X^K, \tau)$ such that the motion is irrotational in $x^{\bar{\alpha}}$, that is,

$$\omega_{\bar{\alpha}\bar{\beta}}(\tau) = 0. \tag{3.11}$$

such a frame exists, see Edelen (1963). Call $x^{\mathring{k}}, x^{\mathring{4}}, \mathring{k} = 1, 2, 3$, a Fermi frame at event point M and let $\Lambda^{\alpha}_{\cdot \mathring{k}=K}(\tau)$ be a Fermi triad at M†. Then, the application of the PMIR (principle of material indifference in relativity) requires that Ψ^* given by (3.1) be written as

$$\Psi^* = \Psi^*(\Lambda_{\bar{\alpha}L} x^{\bar{\alpha}}_{\cdot K}, \Lambda^{\bar{\alpha}}_{\cdot L} \tilde{\mathscr{M}}_{\bar{\alpha}}, \Lambda^{\bar{\alpha}}_{\cdot L} M_{\bar{\alpha}K}, \Lambda^{\bar{\alpha}}_{\cdot K} \mathscr{E}_{\bar{\alpha}}, \theta, \Lambda^{\bar{\alpha}}_{\cdot L} \mathring{\theta}_{\bar{\alpha}}, X^K). \tag{3.12}$$

But it has been shown that (Maugin 1972c)

$$\Lambda_{\bar{\alpha}L}(\tau) = x_{\bar{\alpha}M}(\tau) \mathscr{A}^{M}_{\cdot L}(\tau) \tag{3.13}$$

where $\mathscr{A}^{M}_{\cdot L}$ is a function of $C_{KL}(\tau)$. Hence the equation (3.12) can be written

$$\Psi^* = \Psi^*(\mathscr{A}^{M}_{\cdot L} x_{\bar{\alpha}M} x^{\bar{\alpha}}_{\cdot K}, \mathscr{A}^{M}_{\cdot L} x_{\bar{\alpha}M} \tilde{\mathscr{M}}^{\bar{\alpha}}, \ldots). \tag{3.14}$$

Note that $C_{MK}, M_M, \ldots$ are Lorentz invariants, that is,

$$C_{MK} = x_{\bar{\alpha}M} x^{\bar{\alpha}}_{\cdot K} = x_{\alpha M} x^{\alpha}_{\cdot K}, \qquad M_M = x_{\bar{\alpha}M} \tilde{\mathscr{M}}^{\bar{\alpha}} = x_{\alpha M} \tilde{\mathscr{M}}^{\alpha}, \ldots. \tag{3.15}$$

Thus equation (3.14) reads

$$\Psi^* = \Psi^*(C_{KL}, M_L, M_{LK}, \mathscr{E}_K, \theta, \theta_K, X^K) \tag{3.16}$$

which is equation (3.7). Conversely, one easily checks after some algebra that equation (3.16) is a solution to equation (3.6). Finally, we remark that the set of arguments in equation (3.16) forms a *minimal function basis* of functionally independent arguments, in the sense of Kafadar and Eringen (1971), for Ψ^* expressed by the equation (3.1).

For further use we must compute the proper time rate of Ψ^*. According to equation (3.16), we have

$$\dot{\Psi}^* = \frac{\partial \Psi^*}{\partial C_{KL}} \dot{C}_{KL} + \frac{\partial \Psi^*}{\partial M_L} \dot{M}_L + \frac{\partial \Psi^*}{\partial M_{LK}} \dot{M}_{LK} + \frac{\partial \Psi^*}{\partial \theta} \dot{\theta} + \frac{\partial \Psi^*}{\partial \mathscr{E}_K} \dot{\mathscr{E}}_K + \frac{\partial \Psi^*}{\partial \theta_K} \dot{\theta}_K. \tag{3.17}$$

After equations (3.2) and (3.8 part one) and the definition of the projection operator $P^{\alpha}_{\cdot \gamma}$, we get

$$\begin{aligned} \dot{x}^{\alpha}_{\cdot L} &= \left(u^{\alpha}_{\ ;\gamma} + \frac{1}{c^2} \dot{u}_{\gamma} u^{\alpha} \right) x^{\gamma}_{\cdot L} \\ \dot{C}_{KL} &= 2 x^{\alpha}_{\cdot K} x^{\beta}_{\cdot L} \sigma_{\alpha\beta}. \end{aligned} \tag{3.18}$$

† $\Lambda^{\alpha}_{\cdot \mathring{k}}$ has properties very similar to those of the orthogonal transformation considered in the principle of objectivity of classical continuum mechanics: it is isometric and depends upon the local proper time.

From equations (3.8 part two), (3.18 part one) and (1.6), we obtain

$$\dot{M}_L = \dot{x}^\alpha{}_L \tilde{\mathscr{M}}_\alpha + x^\alpha_{\cdot L}\left(\Omega_{\beta\alpha}\tilde{\mathscr{M}}^\beta + \frac{1}{c^2}\dot{u}_\gamma u_\alpha \tilde{\mathscr{M}}^\gamma\right)$$

$$\dot{M}_L = (u_{\alpha;\gamma} + \Omega_{\alpha\gamma})\tilde{\mathscr{M}}^\alpha x^\gamma_{\cdot L} \tag{3.19}$$

in which we used the fact that $\tilde{\mathscr{M}}_\alpha$ and $x^\alpha_{\cdot L}$ are PU. The computation of $\dot{M}_{LK}$ is more involved. We have

$$\dot{M}_{LK} = A_{LK} + B_{LK} + D_{LK}$$

with

$$A_{LK} \equiv \dot{x}^\gamma_{\cdot L}\tilde{\mathscr{M}}_{\gamma;\mu}x^\mu_{\cdot K}, \qquad B_{LK} \equiv x^\gamma_{\cdot L}\tilde{\mathscr{M}}_{\gamma;\mu}\dot{x}^\mu_{\cdot K}$$

$$D_{LK} \equiv x^\gamma_{\cdot L}\overline{\dot{\tilde{\mathscr{M}}}_{\gamma;\mu}}x^\mu_{\cdot K}.$$

We readily get

$$A_{LK} = \tilde{\mathscr{M}}_{\gamma;\mu}u^\gamma_{\cdot;\rho}x^\rho_{\cdot L}x^\mu_{\cdot K} - \frac{1}{c^2}\dot{u}_\rho u^\gamma_{\cdot;\mu}\tilde{\mathscr{M}}_\gamma x^\rho_{\cdot L}x^\mu_{\cdot K} \tag{3.20}$$

since $\tilde{\mathscr{M}}_\gamma$ is PU. By using (1.6), we get for B_{LK}

$$B_{LK} = x^\gamma_{\cdot L}\tilde{\mathscr{M}}_{\gamma;\mu}u^\mu_{\cdot;\sigma}x^\sigma_{\cdot K} + \frac{1}{c^2}\tilde{\mathscr{M}}^\beta x^\gamma_{\cdot L}x^\sigma_{\cdot K}\Omega_{\beta\gamma}\dot{u}_\sigma. \tag{3.21}$$

In order to compute D_{LK} we note that

$$\overline{\dot{\tilde{\mathscr{M}}}_{\gamma;\mu}} = \tilde{\mathscr{M}}_{\gamma;\mu\nu}u^\nu = \tilde{\mathscr{M}}_{\gamma;\nu\mu}u^\nu = (\tilde{\mathscr{M}}_{\gamma;\nu}u^\nu)_{;\mu} - \tilde{\mathscr{M}}_{\gamma;\nu}u^\nu_{\cdot;\mu} = \dot{\tilde{\mathscr{M}}}_{\gamma;\mu} - \tilde{\mathscr{M}}_{\gamma;\nu}u^\nu_{\cdot;\mu}.$$

But,

$$\dot{\tilde{\mathscr{M}}}_{\gamma;\mu} = \left(\Omega_{\beta\gamma}\tilde{\mathscr{M}}^\beta + \frac{1}{c^2}\dot{u}_\beta u_\gamma \tilde{\mathscr{M}}^\beta\right)_{;\mu}$$

and, performing the differentiation and multiplying the result by $x^\gamma_{\cdot L}x^\mu_{\cdot K}$,

$$x^\gamma_{\cdot L}\dot{\tilde{\mathscr{M}}}_{\gamma;\mu}x^\mu_{\cdot K} = x^\gamma_{\cdot L}x^\mu_{\cdot K}\left(\Omega_{\beta\gamma;\mu}\tilde{\mathscr{M}}^\beta + \Omega_{\beta\gamma}\tilde{\mathscr{M}}^\beta_{\cdot;\mu} + \frac{1}{c^2}u_{\gamma;\mu}\dot{u}_\beta\tilde{\mathscr{M}}^\beta\right).$$

Thus,

$$D_{LK} = x^\gamma_{\cdot L}x^\mu_{\cdot K}\left(\Omega_{\beta\gamma;\mu}\tilde{\mathscr{M}}^\beta + \Omega_{\beta\gamma}\tilde{\mathscr{M}}^\beta_{\cdot;\mu} + \frac{1}{c^2}u_{\gamma;\mu}\dot{u}_\beta\tilde{\mathscr{M}}^\beta - \tilde{\mathscr{M}}_{\gamma;\nu}u^\nu_{\cdot;\mu}\right). \tag{3.22}$$

Collecting the results (3.18) through (3.22), carrying them in equation (3.17), rearranging the indices and introducing when necessary the rate of strain tensor $\sigma_{\alpha\beta}$ and the vorticity tensor $\omega_{\alpha\beta}$ (cf I), we obtain after some lengthy algebra

$$\rho\dot{\Psi}^* = \tilde{t}^{(\beta\alpha)}\sigma_{\alpha\beta} + \tilde{t}^{[\beta\alpha]}(\omega_{\alpha\beta} + \Omega_{\alpha\beta}) + \tilde{M}^{\gamma\beta\mu}A_{\beta\gamma\mu} - \rho\tilde{\eta}\dot{\theta} + \rho\frac{\partial\Psi^*}{\partial\mathscr{E}_K}\dot{\mathscr{E}}_K + \rho\frac{\partial\Psi^*}{\partial\theta_K}\dot{\theta}_K \tag{3.23}$$

in which we have defined, for reasons which shall become clear in the following development, the quantities

$$\tilde{t}^{(\beta\alpha)} = \rho\left(2\frac{\partial\Psi^*}{\partial C_{KL}}x^{(\alpha}_{;K}+\frac{\partial\Psi^*}{\partial M_L}\tilde{\mathcal{M}}^{(\alpha}+\frac{\partial\Psi^*}{\partial M_{LK}}\tilde{\mathcal{M}}^{(\alpha;|\mu|}x_{\mu K}\right)x^{\beta)}_{;L} \tag{3.24}$$

$$\tilde{t}^{[\beta\alpha]} = \rho\left(\frac{\partial\Psi^*}{\partial M_L}\tilde{\mathcal{M}}^{[\alpha}+\frac{\partial\Psi^*}{\partial M_{LK}}\tilde{\mathcal{M}}^{[\alpha;|\mu|}x_{\mu K}\right)x^{\beta]}_{;L} \tag{3.25}$$

$$\tilde{\mathcal{M}}^{\gamma\beta\mu} = \rho\frac{\partial\Psi^*}{\partial M_{LK}}\tilde{\mathcal{M}}^{[\gamma}x^{\beta]}_{;L}x^{\mu}_{;K} \tag{3.26}$$

$$\tilde{\eta} = -\frac{\partial\Psi^*}{\partial\theta}. \tag{3.27}$$

3.3. *Constitutive equations*

In this section, we obtain the recoverable parts of the constitutive equations for a nonlinear (hyper) elastic solid of grade one. Indeed, the constitutive variables $t^{\beta\alpha}$ and $M^{\beta\gamma\alpha}$ present, in general, recoverable and dissipative parts. Hence we write

$$t^{\beta\alpha} = {}^{\mathrm{R}}t^{\beta\alpha}+{}^{\mathrm{D}}t^{\beta\alpha}$$
$$M^{\beta\alpha\gamma} = {}^{\mathrm{R}}M^{\beta\alpha\gamma}+{}^{\mathrm{D}}M^{\beta\alpha\gamma}. \tag{3.28}$$

The conduction current j^γ and the heat flux vector $\hat{q}^\beta$ are, of course, purely dissipative. Then, using the decomposition (3.28), we can write equation (2.25) in the form (2.8), with $\theta > 0$

$$\rho^{\mathrm{R}}\dot{\eta}\theta+\Phi \geqslant 0 \tag{3.29}$$

with

$$\rho^{\mathrm{R}}\dot{\eta} = \theta^{-1}\{-\rho(\dot{\Psi}^*+\eta\dot{\theta})+{}^{\mathrm{R}}t^{(\beta\alpha)}\sigma_{\alpha\beta}+{}^{\mathrm{R}}t^{[\beta\alpha]}(\omega_{\alpha\beta}+\Omega_{\alpha\beta})+{}^{\mathrm{R}}M^{\beta\alpha\gamma}A_{\alpha\beta\gamma}\} \tag{3.30}$$

$$\Phi = \mathscr{E}_\gamma j^\gamma-\frac{1}{\theta}\hat{q}^\beta\overset{*}{\theta}_\beta+{}^{\mathrm{D}}t^{(\beta\alpha)}\sigma_{\alpha\beta}+{}^{\mathrm{D}}t^{[\beta\alpha]}(\omega_{\alpha\beta}+\Omega_{\alpha\beta})+{}^{\mathrm{D}}M^{\beta\alpha\gamma}A_{\alpha\beta\gamma}. \tag{3.31}$$

Carrying the results (3.23) through (3.27) into equation (3.30), we obtain

$$\rho^{\mathrm{R}}\dot{\eta}\theta = -\rho(\eta-\tilde{\eta})\dot{\theta}+({}^{\mathrm{R}}t^{(\beta\alpha)}-\tilde{t}^{(\beta\alpha)})\sigma_{\alpha\beta}+({}^{\mathrm{R}}t^{[\beta\alpha]}-\tilde{t}^{[\beta\alpha]})(\omega_{\alpha\beta}+\Omega_{\alpha\beta})+({}^{\mathrm{R}}M^{\beta\alpha\gamma}-\tilde{M}^{\beta\alpha\gamma})A_{\alpha\beta\gamma}$$
$$-\rho\frac{\partial\Psi^*}{\partial\mathscr{E}_K}\dot{\mathscr{E}}_K-\rho\frac{\partial\Psi^*}{\partial\theta_K}\dot{\theta}_K. \tag{3.32}$$

Note that the quantities in the factors of each rate $\dot{\theta}$, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}+\Omega_{\alpha\beta}$, $A_{\alpha\beta\gamma}$, $\dot{\mathscr{E}}_K$ and $\dot{\theta}_K$ do not depend on the corresponding rates. Thus, for all independent dynamical processes $\dot{\theta}$, $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}+\Omega_{\alpha\beta}$, $A_{\alpha\beta\gamma}$, $\dot{\mathscr{E}}_K$ and $\dot{\theta}_K$, the inequality (3.29) with ${}^{\mathrm{R}}\dot{\eta}$ given by the definition (3.32) can hold if and only if

$$\eta = \tilde{\eta} \tag{3.33}$$

$$\frac{\partial\Psi^*(\mathscr{E}_L)}{\partial\mathscr{E}_K} = 0, \qquad \text{equivalently } \frac{\partial\overline{\Psi}^*(\mathscr{E}_\beta)}{\partial\mathscr{E}_\alpha} = 0 \tag{3.34}$$

$$\frac{\partial\Psi^*(\theta_L)}{\partial\theta_K} = 0, \qquad \text{equivalently } \frac{\partial\overline{\Psi}^*(\overset{*}{\theta}_\beta)}{\partial\overset{*}{\theta}_\alpha} = 0 \tag{3.35}$$

$$ {}^{\mathrm{R}}t^{(\beta\alpha)} = \tilde{t}^{(\beta\alpha)}, \qquad {}^{\mathrm{R}}t^{[\beta\alpha]} = \tilde{t}^{[\beta\alpha]} \tag{3.36} $$

$$ {}^{\mathrm{R}}M^{\beta\alpha\gamma} = \tilde{M}^{\beta\alpha\gamma} \tag{3.37} $$

$$ \Phi \geqslant 0. \tag{3.38} $$

Hence, with

$$ \Psi^* = \Psi^*(C_{KL}, M_L, M_{LK}, \theta, X^K) \tag{3.39} $$

the constitutive equations

$$ \eta = -\frac{\partial\Psi^*}{\partial\theta} \tag{3.40} $$

$$ {}^{\mathrm{R}}t^{\beta\alpha} = \rho\left(2\frac{\partial\Psi^*}{\partial C_{KL}}x^{\alpha}_{;K} + \frac{\partial\Psi^*}{\partial M_L}\tilde{\mathscr{M}}^{\alpha} + \frac{\partial\Psi^*}{\partial M_{LK}}\tilde{\mathscr{M}}^{\alpha}_{;\mu}x^{\mu}_{;K}\right)x^{\beta}_{;L} \tag{3.41} $$

$$ {}^{\mathrm{R}}M^{\beta\alpha\gamma} = \rho\frac{\partial\Psi^*}{\partial M_{LK}}\tilde{\mathscr{M}}^{[\beta}x^{\alpha]}_{;L}x^{\mu}_{;K} \tag{3.42} $$

defines an *inhomogeneous hyperelastic magnetized solid* which is neither a heat nor electricity conductor (the case of incompressible solids is briefly examined in appendix 2). In contrast to the equations derived by Maugin (1971d) in general relativity by use of a variational principle, magnetization gradients are taken into account in the present formulation. This fact is reflected by the presence of the constitutive arguments M_{LK} in the magneto-free energy Ψ^* and the occurrence of the supplementary constitutive variable $M^{\beta\alpha\gamma}$. Equation (3.38) in which the dissipation density Φ is given by (3.31) is the dissipation inequality announced in § 2. The latter inequality we shall use in part III in order to study dissipative processes which result in heat propagation, electrical conduction, dissipative stresses and relaxation of the magnetic spin. Before we must focus our attention on the remarks that follow in the case of saturated magnetization, and show that the field equations and constitutive equations obtained are, for the case of nondissipative processes, in agreement with those derived from a variational treatment (Maugin and Eringen 1972c). This comparison is most important for the equation of conservation of moment of energy–momentum which should take a form closer to that of the classical spin equation (1.10). Note in particular that, if we have obtained a constitutive variable which can represent in a phenomenological way the action of Heisenberg's exchange forces, namely $M^{\beta\alpha\gamma}$, we have not obtained any constitutive variable likely to represent the anisotropy field. In fact, as shown below, this variable is hidden in the formalism used here above.

4. Saturated magnetization

In the case of saturated magnetization, one should be more cautious in performing the differentiation (3.17) for, taking the proper time derivative of M_L and M_{LK} is equivalent to differentiating the magnetization four vector $\tilde{\mathscr{M}}^{\alpha}$ and its gradients $\tilde{\mathscr{M}}^{\alpha}_{;\mu}$ with respect to the proper time. However, as shown hereafter, this fact is irrelevant. Indeed for saturated magnetization, we have

$$ \tilde{\mathscr{M}}^{\alpha}\tilde{\mathscr{M}}_{\alpha} = \tilde{\mathscr{M}}_{\mathrm{S}}^{2} = \text{constant}. \tag{4.1} $$

By differentiation with respect to proper time and space–time coordinates, we get

$$\tilde{\mathcal{M}}^{\alpha}\dot{\tilde{\mathcal{M}}}_{\alpha} = 0 \tag{4.2}$$

$$\tilde{\mathcal{M}}^{\alpha}\tilde{\mathcal{M}}_{\alpha;\mu} = 0. \tag{4.3}$$

According to a remark made in § 1, if we replace $\dot{\tilde{\mathcal{M}}}_{\alpha}$ in the first of these by its value given by equation (1.6), we obtain identically zero. Thus we may disregard the first constraint and need not introduce a Lagrange multiplier. This is to be compared with the fact that the equivalent constraint can be discarded in the classical three-dimensional theory (cf Maugin and Eringen 1972a). Now note that equation (4.3) represents only three independent equations for, multiplying by u^{μ} yields the scalar equation (4.2). Thus, without loss of generality, we can take in lieu of the second constraint

$$\mathscr{L}_K \equiv \tilde{\mathcal{M}}^{\alpha}\tilde{\mathcal{M}}_{\alpha;\mu}x^{\mu}_{.K} = 0, \qquad K = 1, 2, 3 \tag{4.4}$$

which is to be differentiated with respect to proper time if we want to get rates of $\tilde{\mathcal{M}}_{\alpha;\mu}$. We have

$$\dot{\mathscr{L}}_K = \dot{\tilde{\mathcal{M}}}^{\alpha}\tilde{\mathcal{M}}_{\alpha;\mu}x^{\mu}_{.K} + \tilde{\mathcal{M}}^{\alpha}\overline{\tilde{\mathcal{M}}_{\alpha;\mu}}^{\cdot}x^{\mu}_{.K} + \tilde{\mathcal{M}}^{\alpha}\tilde{\mathcal{M}}_{\alpha;\mu}\overline{x^{\mu}_{.K}}^{\cdot}. \tag{4.5}$$

Using equation (1.6) and performing algebra similar to that made in order to get the value of $\dot{M}_{LK}$ in § 3.2, we obtain after some lengthy computations

$$\dot{\mathscr{L}}_K = \tilde{\mathcal{M}}^{\alpha}\tilde{\mathcal{M}}^{\beta}x^{\mu}_{.K}A_{\beta\alpha\mu} = 0. \tag{4.6}$$

This is in fact identically zero since $A_{\beta\alpha\mu}$ is skewsymmetric in the indices β and α. Hence we need not introduce three Lagrange multipliers L^K $(K = 1, 2, 3)$ in order to take account of the constraint (4.6). We can say that the saturation of the magnetization is implied in the formula (1.6). This is indeed clear if we carry on the identification of the magnetization $\tilde{\mathcal{M}}_{\alpha}$ with a *director*. In the paper of Maugin and Eringen (1972d) which considers relativistic oriented continua, an equation similar to (1.6) is given. It describes the motion of a *rigid*, that is, of constant norm, *director*. The equivalent statement here is that the magnetization has a constant modulus in space–time. Hence we may try without any further precaution to bring together the results of § 3 with those of the variational approach which was indeed given for the saturated case.

5. Comparison with a variational approach

The comparison can only be carried out for nondissipative processes since the variational principle was, of course, established with this hypothesis; so we set

$$j^{\gamma} = 0 \qquad (\mathscr{E}^{\gamma} = 0)$$

$$\hat{q}^{\beta} = 0, \qquad {}^{\mathrm{D}}t^{\beta\alpha} = {}^{\mathrm{D}}M^{\beta\alpha\gamma} = 0. \tag{5.1}$$

We thus omit the left superscript R in the remaining constitutive equations. We remark that, given the form of the constitutive equation (3.42), we can introduce in lieu of $M^{\gamma\beta\mu}$ another constitutive variable $\mathfrak{M}^{\beta\mu}$ by

$$M^{\gamma\beta\mu} = \tilde{\mathcal{M}}^{[\gamma}\mathfrak{M}^{\beta]\mu}, \qquad \mathfrak{M}^{\beta\mu} = \rho\frac{\partial\Psi^*}{\partial M_{LK}}x^{\beta}_{.L}x^{\mu}_{.K}. \tag{5.2}$$

$\mathfrak{M}^{\beta\mu}$ is also a PU tensor field as is readily checked. Furthermore it behaves like an axial four vector with respect to its first index. It follows that

$$M^{\gamma\beta\mu}{}_{;\mu} = \tilde{\mathscr{M}}^{[\gamma}\mathfrak{M}^{\beta]\mu}{}_{;\mu} + \rho\frac{\partial\Psi^*}{\partial M_{LK}}\tilde{\mathscr{M}}^{[\gamma}_{\cdot;\mu}x^{\beta]}_{\cdot L}x^{\mu}_{\cdot K}. \tag{5.3}$$

Carrying the constitutive equation (3.36 part two) and the result (5.3) into equation (2.19), we obtain

$$\frac{\rho}{2}\dot{S}^{\alpha\beta} = \rho\tilde{\mathscr{M}}^{[\alpha}\mathscr{B}^{\beta]} + \tilde{\mathscr{M}}^{[\alpha}\mathfrak{M}^{\beta]\mu}{}_{;\mu} + \rho\frac{\partial\Psi^*}{\partial M_L}\tilde{\mathscr{M}}^{[\alpha}x^{\beta]}_{\cdot L} + \frac{1}{c^2}\rho S^{[\alpha}_{\cdot\gamma}u^{\beta]}\dot{u}^{\gamma} - \frac{2}{c^2}\tilde{\mathscr{M}}^{[[\alpha}\mathfrak{M}^{\gamma]|\mu|}u^{\beta]}u_{\gamma;\mu}. \tag{5.4}$$

That is,

$$\left(\frac{\rho}{2}\dot{S}^{\alpha\beta} - \frac{1}{c^2}\rho S^{[\alpha}_{\cdot\gamma}u^{\beta]}\dot{u}^{\gamma}\right) = \rho\tilde{\mathscr{M}}^{[\alpha}\mathscr{B}^{\beta]} + \rho\tilde{\mathscr{M}}^{[\alpha}_{L}\mathscr{B}^{\beta]} + \left(\tilde{\mathscr{M}}^{[\alpha}\mathfrak{M}^{\beta]\mu}{}_{;\mu} - \frac{2}{c^2}\tilde{\mathscr{M}}^{[[\alpha}\mathfrak{M}^{\gamma]\mu}u^{\beta]}u_{\gamma;\mu}\right) \tag{5.5}$$

in which we have introduced the new PU constitutive variable

$${}_L\mathscr{B}^{\alpha} \overset{\text{def}}{=} \frac{\partial\Psi^*}{\partial M_L}x^{\alpha}_{\cdot L}, \qquad {}_L\mathscr{B}^{\alpha}u_{\alpha} = 0 \tag{5.6}$$

which we call *local field* or, better, *anisotropy field* since further identification shows that this is the phenomenological role we can assign to this quantity. It is readily verified that the left hand side of equation (5.5) is nothing but the projection of $\frac{1}{2}\rho\dot{S}^{\alpha\beta}$ on to the hypersurface $M^3_{\perp}$ (cf notations of part I) while the last term in the right hand side of this equation is the projection of the quantity $\tilde{\mathscr{M}}^{[\alpha}\mathfrak{M}^{\beta]\mu}{}_{;\mu}$ on to $M^3_{\perp}$. This is proved by playing with the skewsymmetries in the last term within parentheses. Finally, recalling that (cf equation (I-3.40))

$$S^{\alpha\beta} = \gamma^{-1}\tilde{\Pi}^{\alpha\beta} \tag{5.7}$$

in which γ is the gyromagnetic ratio of an electron and $\tilde{\Pi}^{\alpha\beta}$ is the magnetization two-form per unit of proper mass, and noting that $\mathscr{B}^{\alpha}, {}_L\mathscr{B}^{\alpha}$ and $\tilde{\mathscr{M}}^{\beta}$ are PU, then for example,

$$\mathscr{B}^{[\alpha}\tilde{\mathscr{M}}^{\beta]} \equiv P^{\alpha}_{\cdot\gamma}P^{\beta}_{\cdot\delta}\mathscr{B}^{[\gamma}\tilde{\mathscr{M}}^{\delta]}$$

we can write equation (5.5) as

$$\mathbf{P}\{\dot{\tilde{\Pi}}^{\alpha\beta}\} = \mathbf{P}\{2\gamma\tilde{\mathscr{M}}^{[\alpha}\mathscr{B}^{\beta]}_{\text{eff}}\} \tag{5.8}$$

where the *effective field* $\mathscr{B}^{\alpha}_{\text{eff}}$, a PU four vector field, is defined as

$$\mathscr{B}^{\alpha}_{\text{eff}} \equiv \mathscr{B}^{\alpha} + {}_L\mathscr{B}^{\alpha} + \rho^{-1}\mathfrak{M}^{\alpha\mu}{}_{;\mu} \tag{5.9}$$

and we have used the short hand notation $\mathbf{P}$ for the operation of projection on to $M^3_{\perp}$. Equation (5.8) is the form forecast in Maugin and Eringen (1972a) (cf equation (5.14) of that paper and the remark thereof) but it is relativistically invariant. It represents three independent field equations.

At least for nondissipative processes, the system of field equations obtained in part I is now in closed form. Though we introduced new constitutive variables ${}_L\mathscr{B}^{\alpha}$ and $\mathfrak{M}^{\alpha\mu}$, the number of unknowns is unchanged, for $t^{(\beta\alpha)}$, ${}_L\mathscr{B}^{\alpha}$ and $\mathfrak{M}^{\alpha\mu}$ on the one hand and $t^{\beta\alpha}$ and $M^{\beta\alpha\gamma}$ on the other have exactly the same number of independent components.

Recall now that in the variational approach (Maugin and Eringen 1972c) which used two-form (skewsymmetric tensor) formalism and not four vector notation, the magnetic spin equation obtained (not projected on to $M^3_\perp$) was

$$\dot{\tilde{\Pi}}^{\alpha\beta} = 2\gamma \overset{*}{F}{}^{[\alpha}_{\cdot\mu}\tilde{\Pi}^{|\mu|\beta]} \tag{5.10}$$

in which the *effective magnetic flux* tensor $\overset{*}{F}{}^{\alpha\gamma}$ (equivalently the rotation of $\tilde{\mathscr{M}}^{\beta}$ since $\Omega^{\alpha\beta} = \gamma\overset{*}{F}{}^{\alpha\beta}$) was defined as

$$\overset{*}{F}{}^{\alpha\mu} = F^{\alpha\mu} + 2\gamma^{-1}a^{[\alpha}u^{\mu]} + \mathscr{F}^{\alpha\mu} + 2\rho^{-1}\mathfrak{M}^{\alpha\mu\rho}{}_{;\rho}. \tag{5.11}$$

Here $F^{\alpha\mu}$ is the maxwellian magnetic flux tensor and $\mathscr{F}^{\alpha\mu}$ is the *local magnetic flux* or *anisotropy field* tensor. $\mathfrak{M}^{\alpha\mu\rho}$ is the *electromagnetic* hyperstress tensor (skewsymmetric in its two first indices and completely PU). The tensors $\mathscr{F}^{\alpha\mu}$ and $\mathfrak{M}^{\alpha\mu\rho}$ were shown to provide a phenomenological representation of the magnetic spin–crystal lattice and spin–spin interactions respectively. The PU four vector field a^{α} was introduced as a Lagrange multiplier with a view to take account of the Frenkel condition (cf part I) and was shown to be a complicated expression which involves the Lorentz force, $\mathscr{F}^{\alpha\mu}$ and the divergence of $\mathfrak{M}^{\alpha\mu\rho}$. The factor two in front of the divergence of $\mathfrak{M}^{\alpha\mu\rho}$ in equation (5.11) appears because the tensor $\mathfrak{M}^{\alpha\mu\rho}$ has been obtained by differentiating the free energy density with respect to a skewsymmetric tensor but this skewsymmetry has not been taken into account in the process.

Equation (5.11) written in the two-form formalism has, apart from the term involving a^{α}, a structure similar to that of equation (5.9). We are easily convinced that equations (5.11) and (5.9) are dual formulae if we remark that the dual of the skew tensor $a^{[\alpha}u^{\mu]}$ is indeed zero since it would be defined as

$$\overset{*}{a}_{\gamma} = \frac{1}{2ic}\epsilon_{\gamma\alpha\mu\nu}a^{\alpha}u^{\mu}u^{\nu} \equiv 0 \tag{5.12}$$

from the skewsymmetry of $\epsilon_{\gamma\alpha\mu\nu}$.

In absence of electric field (cf equation (5.1)), polarization and polarization gradients, the skewsymmetric tensors $F^{\alpha\mu}$, $\mathscr{F}^{\alpha\mu}$ and $\mathfrak{M}^{\alpha\mu\rho}$ are indeed equivalent to the axial four vectors $\mathscr{B}^{\alpha}$ and ${}_{L}\mathscr{B}^{\alpha}$ and the tensor $\mathfrak{M}^{\alpha\rho}$ (axial with respect to the index α) respectively. The magnetic spin equation and the constitutive equations of the variables which appear in it assume therefore the same form in both approaches. Furthermore the equation (5.10) and now, equation (5.8), have a form similar to those of equations (I-5.8) and (I-5.10) (postulated *a priori*) when the last terms in the right hand sides of these equations vanish. In fact these supplementary terms result from linear but physically involved dissipative processes the study of which will be given in a forthcoming paper. As a final remark, it is also verified that, for small velocities and quasi-magnetostatics, we obtain the equations of the classical three-dimensional theory (cf Maugin and Eringen 1972a).

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Appendix 1

Many authors have emphasized that the magnetic spin gives rise to a couple of the gyroscopic type. A peculiar characteristic of such a couple is that it produces no work in a real displacement (here rotation) field although it does in a virtual displacement field. This fact is linked to the 'kinematical' or *nonholonomic* (ie nonintegrable) nature of the angular velocity field. In the classical three-dimensional theory, since (cf equation (1.1))

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\pi} \times \boldsymbol{\mu} \tag{A.1}$$

one of course has

$$\dot{\boldsymbol{\mu}} \cdot \boldsymbol{\pi} = (\boldsymbol{\pi} \times \boldsymbol{\mu}) \cdot \boldsymbol{\pi} = 0. \tag{A.2}$$

In four-dimensional formalism, we have (cf equations (1.8), (1.9), (5.7) and (I-3.38))

$$\Omega_{\beta\alpha} = \frac{1}{\mathrm{i}c}\epsilon_{\beta\alpha\mu\nu}\pi^{\mu}u^{\nu}, \qquad \pi^{\mu}u_{\mu} = 0 \tag{A.3}$$

$$\dot{\tilde{\mathscr{M}}}^{\alpha} = \frac{1}{\mathrm{i}c}\epsilon^{\alpha\mu\sigma\nu}\pi_{\mu}\tilde{\mathscr{M}}_{\sigma}u_{\nu} + \frac{1}{c^2}\dot{u}_{\mu}\tilde{\mathscr{M}}^{\mu}u^{\alpha} \tag{A.4}$$

$$S^{\alpha\beta} = \frac{\gamma^{-1}}{\mathrm{i}c}\epsilon^{\alpha\beta\gamma\delta}\tilde{\mathscr{M}}_{\gamma}u_{\delta}, \qquad \tilde{\mathscr{M}}_{\gamma}u^{\gamma} = 0. \tag{A.5}$$

Then, using the two last equations, the real work of the magnetic spin is

$$A \equiv \frac{1}{2}\dot{S}^{\alpha\beta}\Omega_{\beta\alpha} = \frac{\gamma^{-1}}{\mathrm{i}c}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\mu\nu}\overline{\tilde{\mathscr{M}}_{\gamma}u_{\delta}}\pi^{\mu}u^{\nu} \tag{A.6}$$

but,

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\mu\nu} = 2(\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu} - \delta^{\gamma}_{\nu}\delta^{\delta}_{\mu}) \tag{A.7}$$

so that equation (A.6) reads

$$A = -\gamma^{-1}\dot{\tilde{\mathscr{M}}}_{\mu}\pi^{\mu} \tag{A.8}$$

in which we have used (A.3 part two), (A.5 part two) and the equations (cf part I)

$$u_{\nu}u^{\nu} = -c^2, \qquad \dot{u}_{\nu}u^{\nu} = 0. \tag{A.9}$$

Finally, using (A.4), we obtain

$$A = -\gamma^{-1}\left(\frac{1}{\mathrm{i}c}\epsilon_{\mu\delta\sigma\nu}\pi^{\delta}\tilde{\mathscr{M}}^{\sigma}u^{\nu} + \frac{1}{c^2}\dot{u}_{\nu}\tilde{\mathscr{M}}^{\nu}u_{\mu}\right)\pi^{\mu} = 0 \tag{A.10}$$

from the skewsymmetry of $\epsilon_{\mu\delta\sigma\nu}$ and equation (A.3 part two).

Appendix 2. Incompressibility

A relativistic incompressible solid may be defined as one for which the following condition holds:

$$\Theta = \sigma^{\alpha}_{\cdot\alpha} = 0. \tag{A.11}$$

That is, the dilatation vanishes. According to the continuity equation (I-4.1), the invariant relativistic density ρ is then constant along a world line. Since $\sigma_{\alpha\beta}$ is a PU tensor, (A.11) can be written as

$$P^{\alpha}_{\cdot\beta}\sigma^{\beta}_{\cdot\alpha} = 0. \tag{A.12}$$

This constraint must be taken into account in performing the differentiation (3.17). Introducing a Lagrange multiplier p referred to as the *mechanical pressure* (an unknown to be determined upon solving a peculiar well posed problem with given boundary conditions and initial conditions on a space-like hypersurface), we should write the left hand side of equation (3.23) as $\rho\Psi^* + pP^{\alpha\beta}\sigma_{\alpha\beta}$. This would result in adding a term $-pP^{\alpha\beta}$ to the constitutive equation of $t^{\beta\alpha}$.

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